Are numbers' mental representations spatially encoded? Around the representational Effect in Mathematics

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Abstract | Current cognitive accounts on numbers' mental representations do not agree on their spatial-sensitive nature. On one hand, a Internalist neuro-based approach, supported by trans-notational conservation of cognitive effects related with mathematical tasks has postulate a *quantity-based analogical format* while the concurrent Externalist problem solving based approach has suggested that some of the primitive components of numbers' mental representation might depend directly to the spatial features of numbers' written representations. The purpose of this article is an epistemological and descriptive one. Advocating for the second tradition, we shall present both accounts, focusing on exotic cases of non-positional number systems, to disentangle their shallow contradiction.

Keywords | Mental calculation, numbers systems, spatial reasoning, representational effect

Introduction

This article do not have anything to do with numbers *per se* as numbers theory does nor is it directly concerned with history of algebra. The main purpose of this paper is to investigate the impact of numeration on effective calculus realized by a finite automaton; namely the human mind. Even if this view sounds very subjective or solipsist, numerous historical approaches such as the formalist (Hilbert and Ackermann 1928), intuitionist (Brouwer 1992a,b,c) or algorithmic schools (Kolmogorov and Uspenskii 1958) endorsed what became a classical idea in theory of demonstration following which a proof – thus a calculus – is a finite number of formal operations from some syntactic object (premices) to an other (result).

The reader may wonder what such a view has to do with daily calculus. The first indication is an introspective one: the fact that, for some complex operations, subjects report doing the calculus as if they were writing it on a mental board. The second comes from historical experimental results which suggest "[...] *that cognitive operations are not independent of the symbols that instigate them and that "information" is not wholly separable from its embodiment in a symbol system.* " (Gonzalez and Kolers 1982:319). In the same idea, many comparative analysis suggest that European and Asian child populations do not apprehend place values of the Hindu-Arabic system in the same fashion (Miura et al. 1993) because of some practices such as hand counting (Domahs et al. 2010). Finally, it should be stressed that this issue differs from the influence of surface features on mathematical problem solving – see Brissiaud and Sander 2010 for an example – for it is concerned with the very influence of numbers physical features themselves rather than the bias induced by the formulation of a given concrete problem as Becker and Varelas 1993 distinguished it.

We will first described the main features of non-positional numbers systems –sec.1– to present two very different historical frameworks in sec.2 and sec.3 opposition of which concerns the very status of external representations of numbers toward internal ones. In sec.4, the reasons why these two cognitive accounts did not discuss before recently will be highlighted. With the help of current research on the subject, we will suggest that, in the absence of precision of the kind of mathematical tasks, these two frameworks do not contradict each other and could well be complementary.

1 What is a (non)standard number system: a syntactic account

The easiest way for suggesting what a non Standard Positional Number System (n-SPNS) is consists in suggesting the common properties of Standard Positional Number System (SPNS).

Standard Positional Number System are the most used systems in the modern world. The 10-based Hindu-Arabic system –eq.1– is only one way to express numbers for it is possible to convert it in 7-based –eq.2–, 2-based –eq.3. The main property of a positional number system lies in the notion of *digit* provided that every single number can be, for a given base b, decomposed by the order of its digits (d) in the polynomial way presented in eq.4.

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10... \tag{1}$$

$$1, 2, 3, 4, 5, 6, 7, 10, 11, 12... \tag{2}$$

$$0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010... \tag{3}$$

$$d_n * b^n + d_{n-1} * b^{n-1} + d_{n-2} * b^{n-2} + ... + d_1 * b + d_0 \tag{4}$$

In fact, as eq.5 suggests it, one can get rid of Arabic digits provided that she has an ordered set of symbols – e.g. ♣, ♠, ♦. Most of the symbols sets used in computer sciences are also, no matter how large, ordinate ones as the well known hexadecimal coding (16-based: 0-F).

$$\clubsuit, \spadesuit, \blacklozenge, \spadesuit\clubsuit, \spadesuit\spadesuit, \spadesuit\blacklozenge, \blacklozenge\clubsuit, \blacklozenge\spadesuit, \blacklozenge\blacklozenge, \spadesuit\clubsuit\clubsuit, \spadesuit\clubsuit\spadesuit... \tag{5}$$

non-Standard Positional Number System lack one of the properties of SPNS. Even if some systems are merely non-positional – we can imagine, for instance, an unstructured extensional system in which one add only "■" for every new object – most of the n-SPNS remain syntactically ordoned on one dimension. Though, the base b can be negative (Kempner 1936), real (Frougny 1992) or even a complex number as the Quater-Imaginary Numeral system of Knuth 1960. An other family of n-SPNS combines two bases such as the Babylonian Sexagesimal system ($b_1 = 60$ and $b_2 = 10$) (Powell 1976). Finally, the non-standardness of some systems such as the Balanced Ternary –eq.6– or the set-theoretic way of writing numbers –eq.7– seems more dependent of syntactical features; more *graphical* so to speak.

$$0, 1, 1T, 10, 11, 1TT, 1T0, 1T1, 10T, 100, 101 \tag{6}$$

$$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\}, \{\emptyset, \{\emptyset, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\}\}, \tag{7}$$

$$\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset\}\}\}\}\}\},\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset\}\}\}\}\}\}\},$$

$$\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset\}\}\}\}\}\}\}\},\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset\}\}\}\}\}\}\}\}\},$$

$$\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset\}\}\}\}\}\}\}\}\}\}$$

One may say that these differences concern only bases or writing. We would like to show that some of the features of n-SPNS have a direct impact on the syntactical aspect of numbers on a deeper level than purely graphical criterion. Despite the fact that n-SPNS remain syntactically ordoned on one dimension –i,e. written on a structured line– the syntactical rules used for, say, add 1 seem more complex than those used in SPNS such as the Hindu-Arabic system. Given that written numbers are strings of characters, it is possible to describe theses numbers as products of finite state automatons –i,e. Turing Machines [2]– similar to those used in generative grammars [3].

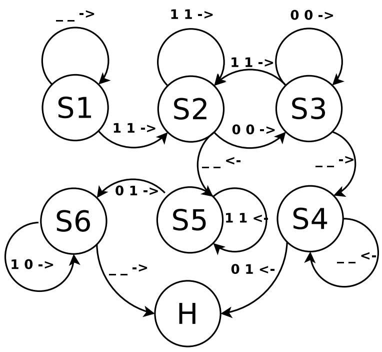

Figure 1: Generative grammar of Binary Number System

As an illustration, tab.1 compare two famous SPNS (resp n-SPNS): strictly positive binary notation and a subset of Roman number system. Here we need to highlight two points. Firstly, even if the length of both set of symbols does not vary much –3 Vs 4– n-SPNS is grammatically much more complex than the binary system. However, this complexity doesn't have much to do with the number of state

[2]Many Turing machines presentation rely on a 4-uplet syntax which distinguishes *move action* from *writing action*. However, given i) the functional equivalence and ii) the length of instruction list in 4-uplet syntax, we will adopt here a 5-uplet syntax $< State_{actual}, Symbole_{read}, State_{next}, Symbole_{written}, move >$. Here, _ will be the blank symbol and H, the "*Halt*" state.

[3]We will restrain our example to the Roman Number system to sketch one of the more striking syntactical features of n-SPNS instead of demonstrating the complete set of properties of all variations. For a more exhaustive description of generative grammars of n-SPNS, see Frougny 1992.

	Binary system	**Roman system**
Set of symbols	0,1,$\_$	I, V, X, $\_$
List from 1 to 7	$1, 10, 11, 100, 101,$ $110, 111, 1000, 1001$	$I, II, III, IV, V,$ $VI, VII, VIII, IX$
Turing machine implementation	$< S_1, \_, S_1, \_, \rightarrow>$ $< S_2, \_, S_5, \_, \leftarrow>$ $< S_1, 1, S_2, 1, \rightarrow>$ $< S_5, 1, S_5, 1, \leftarrow>$ $< S_2, 1, S_2, 1, \rightarrow>$ $< S_5, 0, S_6, 1, \rightarrow>$ $< S_2, 0, S_3, 0, \rightarrow>$ $< S_6, 1, S_6, 0, \rightarrow>$ $< S_3, 0, S_3, 0, \rightarrow>$ $< S_6, \_, H, \_, \rightarrow>$ $< S_3, 1, S_2, 1, \rightarrow>$ $< S_3, \_, S_4, \_, \rightarrow>$ $< S_4, \_, S_4, \_, \leftarrow>$ $< S_4, 0, H, 1, \leftarrow>$	$< S_1, \_, S_1, \_, \rightarrow>$ $< S_4, V, S_6, \_, \rightarrow>$ $< S_1, I, S_2, I, \rightarrow>$ $< S_6, \_, S_7, I, \rightarrow>$ $< S_2, I, S_3, I, \rightarrow>$ $< S_7, \_, H, X, \rightarrow>$ $< S_2, \_, H, I, \rightarrow>$ $< S_1, V, S_9, V, \rightarrow>$ $< S_3, I, S_4, \_, \leftarrow>$ $< S_9, I, S_2, I, \rightarrow>$ $< S_3, \_, H, I, \rightarrow>$ $< S_9, \_, H, I, \rightarrow>$ $< S_4, I, S_4, \_, \leftarrow>$ $< S_2, V, S_8, V, \leftarrow>$ $< S_4, \_, S_5, I, \rightarrow>$ $< S_8, I, H, \_, \rightarrow>$ $< S_5, \_, H, V, \rightarrow>$
Diagram	fig.1	fig.2

Table 1: Comparison of the complexity of to well known SPNS and n-SPNS

–6 Vs 8– but with transitional complexity. As fig.1 and fig.2 suggest it, we do not have the same reflexive and symmetric loop able to scan the whole string before effecting writing changes –see S_2 and S_3 relations in fig.1.

A contrario, the generative grammar responsible for Roman Number System needs to multiply some apparently identical states to count the length of some sections of the string to implement rules such as ”$III + I = IV$”–see S_2, S_3 and S_4 in fig.2. The reason why is that n-SPNS are different from place-valued SPNS; the most relevant syntactical feature of n-SPNS is the length of some sub-strings and the fact that their position in the whole string matter. In other words, generative grammar of n-SPNS such as Roman Number System are string length sensitive.

This first syntactical account of SPNS and n-SPNS stress that the logic of the later and the graphical component which implement it are very different than those of the former. It is also clear that these differences do not have much to do with the particular shape of the symbols but rather with some more fundamental features of the number system considered.

Though, despite the intrinsic quality of the traditional Turing-based paradigm and its relevance in lots of cognitive fields such as formal linguistics, it would be naïve to think that an actual human being really watches every single symbol till the end of the string before making linear transformations as in our example. Even if the syntactical description remains one of the best way to highlight the main differences between SPNS and n-SPNS, the cognitive aspect of calculation is not apparent yet in such a framework.

2 Internal Representations of Numbers: a neural code for mathematics

History did not wait for the 90' to propose neural correlate of mathematical ability. Gall and Spurzheim 1810's *phrenology*, one of the most famous account of the last century, postulated a special area dedicated to calculation in the temporal lobe – near the constructiveness area. However, despite the general discredit of this macro-scale neural instantiation, more parsimonious proposals emerged during the 90'. Rather than speaking of a general ability for mathematics, whatever this can mean, modern studies focus on the possible neural correlate of some mathematical task – especially equation solving in arithmetic – with a theoretical framework which distinguishes different levels of processing from a recognitionnal number extraction to computation.

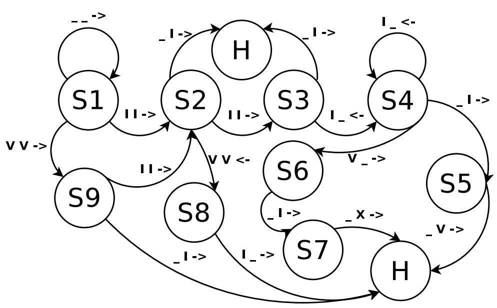

Figure 2: Generative grammar of Roman Number System

2.1 A new perspective on this old problem

We will focus mainly on the review presented in Dehaene, Dehaene-Lambertz, and Cohen 1998 whose formula constitute the title of this subsection. The general thesis of this article consists in suggesting a common ability to extract numerosity from stimuli and compute these cardinal numbers across species. One of the main difficulty with such a generic object consists in the fact that experimenters who have access to merely non-numerical behaviours –e.g. a Rhesus monkey or a two months-old children would not say anything about numbers but, rather, choose some set faster among others – need to identify genuine numerical representations.

Number of what? According to Dehaene, Dehaene-Lambertz, and Cohen 1998, the best functional criterion for identifying genuine numbers representation lies in the notion of *cardinality*. As an abstract notion, cardinality refers to "*the number of things with belong to the same set*". As a cognitive one, a subject can be said to posses such general and extentionnal scheme if he is able to perform operations which presuppose to get rid of the particularities of sets –e.g. sizes, colors, shapes– of modalities –e.g. earing three sounds vs seeing three balls– or modes of presentation –simultaneous vs sequential.

The case of children and animals As behavioural results testifying for such an abstract *number sense* (Dehaene 1999), newborn have been shown, while accustomed to see a certain number of object on the screen, to gaze for a longer time the different slide (Antell and Keating 1983). Even though one could argue that this finding does not account for an abstract notion of number, precautions such as variations of objects (Strauss and Curtis 1981) and partial occlusion of the field (Van Loosbroek and Smitsman 1990) have been taken to avoid merely low-level visual effects. Besides visual induced numerosity, children seem also able to recognize matching across modalities (Starkey, E. S. Spelke, and Gelman 1983, 1990) and to perform basic numerical operations (Koechlin 1997; Wynn 1992).

Even in the animal kingdom, similar numerosity identifications and basic operation have been demonstrated for rats (Meck and Church 1983). Besides the Arabic digit recognition and use (Matsuzawa 1985), chimpanzees seem also able to perform simple operations (Pérusse and Rumbaugh 1990).

2.2 What role does the external world play ?

Numerical distance and Number size effects An other group of experimental studies suggests that animal and human representations of numerosity share a common feature which make them sensitive to distance effect on rate of error in comparison tasks. In other word, both humans (Gallistel and Gelman 1992) and animals (Washburn and Rumbaugh 1991) tend to make less error in comparing ● ● ● ● ● ● ●● with ●● than ● ● ● ● ●● with ● ● ● ● ●. Such an exponential decrease of errors over distance is not retricted to set comparison for similar results has been found in the human case for Arabic digits comparison (Dehaene, Dupoux, and Mehler 1990) and even for for numbers below 10 (Dehaene 1996). Likewise, a similar effect on error does not depend on distance between numbers but seems size-depended inasmuch as comparison of 3 and 4 appears faster than 8 with 9 even for trained adults with dots (Van Oeffelen and Vos 1982) or Arabic and written numbers.

> The occurrence of a distance effect even when numbers are presented in a symbolic notation suggests that the human brain converts numbers inter-

> nally from the symbolic format to a continuous, quantity-based analogical format. (Dehaene, Dehaene-Lambertz, and Cohen 1998:358)

Abstract numerical representations It should be emphasize that these two effects support the internalist conception of number representations endorsed by these authors. In a nutshell, this conception of representation is intrinsically related with the postulation of a notational-independent system for *deep semantic analysis* which has been proposed to explain high-level case of unconscious priming effect independent from notations (e.g. ”TWO” vs ”2” see Dehaene, Cohen, et al. 2005; Dehaene, Naccache, et al. 1998). Notation independence is also the common finding of many studies which show that Reaction Times in number comparison tasks remain the same across Arabic (eg.2), number words (eg.TWO), roman numerals (eg. II) and sets of objects (eg.●●) (Barth, Kanwisher, and E. Spelke 2003; Barth, La Mont, et al. 2005; Dehaene and Akhavein 1995; Schwarz and Ischebeck 2000) which is supported by many brain imaging studies (Eger et al. 2003; Naccache and Dehaene 2001; Pinel et al. 2001) – see Verguts and Fias 2004 for a computational model of non-symbolic encoding of symbols inspired by this literature and Herrera and Macizo 2008 for an experimental effect consistent with this model.

Although we will try to disentangle these questions in sec.4, this theoretical challenge has also to do with the detraction of older symbolic-based models of cognition. Here are the reasons why they assume that ”*this internal access to quantity seems to be a compulsory step in number processing* [...]” (Dehaene, Dehaene-Lambertz, and Cohen 1998:358)

Dissociated abilities However, this commitment in distinguishing an encoding section before any computation puts under new lights selective deficits of calculation. As an interesting clinical case, Mr M (Dehaene and Cohen 1997) suffers from dyscalculia which makes him unable to perform simple one digit subtractions such as 3-1 (75% of errors), magnitude comparison (14% of errors) or indicating which number fell in the middle of two others (77%) while analogous problems about letters or days of the week remain intact. Interestingly, addition and multiplication tasks remain also intact because of the verbal memorization of tables during childhood (e.g. "*6 time 4, 24*").

Similar dissociations of verbal and memorized components of maths –such as tables– from computational counterparts have also been suggested for other kind of lesions (Takayama et al. 1994) and present interesting parallels with the results of other fields of cognitive sciences which advocate for a verbal routine system different from a symbolic and computational one (T. N. Carraher, D. W. Carraher, and Schliemann 1987).

Neural imagery These clinical finding are consistent with the brain-imaging results summarized by Dehaene, Piazza, et al. 2003 – see tab.2 – which suggest that some of the primary components of calculation are distributed in three distinct regions of the occipital part of the brain. First, the horizontal segment of the intraparietal sulcus (HIPS) seems sensitive numerical quantities manipulations which cannot be reduced to rote learning of tables. In contrast, verbally oriented tasks such as multiplication and addition favour the Angular gyrus (AG). Finally, given that the posterior superior parietal lobule (PSPL) is activated during number comparison, approximation during harder tasks, subtraction of two digits, counting (Piazza

TASKS	HIPS	AG	PSPL
Comparison of one-digit numbers vs letter naming	Chochon et al. 1999		
Subtraction of one digit from 11 vs comparison of one-digit numbers	Chochon et al. 1999		
Subtraction of one digit from 11 vs letter naming	Lee 2001		
Subtraction of one-digit numbers from 11 **vs** letter naming	Simon et al. 2002		
Subtraction vs multiplication of one-digit numbers			Lee 2001
Numerosity **estimation vs** physical matching	Piazza et al. 2002b		
Distance effect in comparison of two-digit numbers	Pinel et al. 2001		Pinel et al. 2001
Size effect in exact addition of one-digit numbers	Stanescu-Cosson et al. 2000		
Inverse **size effect** in exact addition of one-digit numbers		Stanescu-Cosson et al. 2000	
Multiplication vs comparison of one-digit numbers		Chochon et al. 1999	
Multiplication vs subtraction of one-digit numbers		Lee 2001	
Intersection of subtraction and phoneme detection tasks		Simon et al. 2002	
Approximate vs exact addition of one-digit numbers			Dehaene, E. Spelke, et al. 1999

Table 2: Three regions involved in various mathematical tasks: summary of fMRI results presented in Dehaene, Piazza, et al. 2003. "**vs**" indicates that results have been obtained by contrasting two tasks. Bold font indicate the relevant task.

et al. 2002a) and dual operations (Menon et al. 2000), we may consider that it acts like an attentional system dedicated to numbers which manage resources for the different operations.

3 Distributed Representations: a theoretical framework

Zhang and Norman 1995's account for the cognitive dimension of mathematics is far different from the one presented in sec.2 for it is mainly concerned with the *representational effect* of number systems – see McCloskey and Macaruso 1995 for a contemporaneous taxonomy of forms of numerical representations. The reason why has to do with Zhang's general thesis following which human problem solving is massively supported by external representations of the given problems. We will present these general ideas in sec.3.1 before coming back on number systems in sec.3.2.

3.1 Representational Effects as a General Framework for Problem Solving

As pointed out in Zhang 1997, the fact that problem solving relies on the external representation of a given problem can be understood in two ways. The first, similar to the theory presented in sec.2, consists in considering external representations merely as a memory support similar to a Turing-machine tape, so to speak. In a nutshell, the notion of *encoding* is central for this traditional view for it considers problem solving as if the subject were i) encoding the visual data of a given problem – say, ”3 + 4” – into a completely different internal representation of which she ii) performs her operation before iii) re-encoding the answer– say ”7”.

The alternative view on external representations – initiated by Gibson 1966, 1979 – states that subjects exploit the ”*highly structured-full of invariant information in the extended spatial and temporal patterns of optic arrays*” (Zhang 1997:181); i.e visual environment. Moreover, it suggests that such a use of the spatial or structural features of external stimuli does not suppose any use of the memory but rather a simple pick up of the relevant environmental information for the given problem. Thus, such a view advocates that the cognitive scientist interested in problem solving and, especially, mathematics has to deal with both internal, external representations and the relation they entertain.

> The key assumption of the framework is that external representations need not be re-represented as an internal model in order to be involved in problem solving activities: they can directly activate perceptual operations and directly provide perceptual information that, in conjunction with the memorial information and cognitive operations provided by internal representations, determine problem solving behavior. (Zhang 1997:187)

ER-based problem: an illustration As in analogy-based theories of problem-solving (Holyoak 1984, 1990; Holyoak and Bower 1985; Holyoak, Junn, and Billman 1984; Holyoak and Thagard 1989), the notion of *isomorphism* between structures of problems or representations is a central one. Though, ER-based theories such as Norman 2010 explicitly focus on the relations between internal and external representations in distinguishing these two components from Abstract Task Structures which are most of the time unavailable to task performers.

The experiment presented in Zhang 1997 consists in performance comparison in three isomorphic problems of the Tic-Tac-Toe (TTT) – see fig.3 – goal of which consists in aligning first three X vertically, horizontally or diagonally – resp. O – in a 3x3 grid-pattern to win the game. The *number* version – see fig.3 (B) – consists in choosing number sum of which is equal to 15. The *shape* variation – see fig.3 (C) – consists in choosing three circles that contain a common shape. Finally, the *color* problem – not represented on fig.3 – consists in getting three circles that contain a

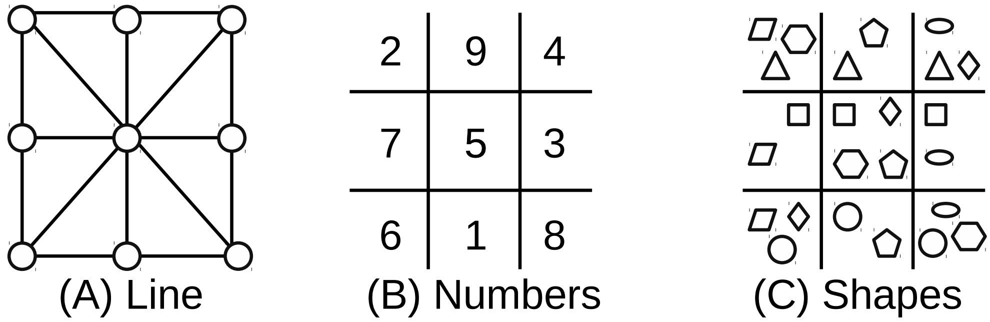

Figure 3: Isomorphism of Abstract Task Structures

common color. We should stress that circles are not positioned on a grid pattern in tasks B, C and D.

The reader may raise doubt about the isomorphism of such variations. However beyond appearance, this three problems are isomorphic for every single functional aspect of the original TTT has an equivalent in B and C – see fig.3. In this way, 5, the 4 shapes circle and the 4 colors circle are strictly equivalent to the center in the original TTT inasmuch as they can be used in the 4 *Winning Triplets*. In a nutshell, we also have the same *Symmetry Categories*, and *Winning Invariants* as in the original TTT game.

However, this isomorphism concerns only *Abstract Task Structures* for there is a fundamental difference between these problems on a representational level. For instance, the only direct perceptual operation in the *number* variant consists in identifying circles. Thus the only external representation in such a variation is digit segmentation while all the relevant aspects of the task are encoded in internal representations of numerical values, sums of digits and parity. In contrast, all the data relevant to perform the *Shape* task such as i) circles ii) common shape circle or iii) quantity of objects are directly observable. Thus, the only difference between the *Shape* task and the original TTT task is that the directly perceived information of the former – a.k.a. *Quantity of objects* – may raise a ”*more-is-better*” bias different from the ”*central-is-better*” bias of the later.

Results presented in Zhang 1997: 195–201 suggest that the disparity of performances across tasks does not have merely to do with ”*more-is-better*” kind of biases though the *Number* variation suffers from the worst constraint. Indeed, performances across *Color* and *Shape* variations also differ though these tasks seem the same. In a nutshell the reason why is related with the fact that the external representation in the *Shape* variant underlines the discreteness of the physical entities which have to be associated whereas this aspects is less striking in the *Color* task.

As a control experiment for his claim, Zhang also presented an other set of TTT non-aligned variants which distinguish two key features, namely, task – TTT or *Color* tasks – and a symmetrical vs asymmetrical conditions. The results present very similar patterns of TTT and *Color* task for the asymmetrical condition both for i) the first move, ii) rate of unsolved problem – i.e. 9 moves without wining against the computer – and iii) rate of solved problems – above random. This finding strongly suggests that external symmetry plays a critical role in suggesting the *Symmetry Categories* of an *Abstract Task Structure*. To sum up, some visual features may or may not represent the structural properties of a given problem in a adequate fashion. In case of inadequacy, subjects rely massively on internal representations without any direct and immediate external ones which explain the biases.

3.2 Numbers, dimensions and external representations

Despite the singularity of numbers systems in cognitive activities, the general framework presented above has been proposed as an analysis for numbers systems in Zhang and Norman 1995. Like any other problem-solving, a calculus relies on external representations of the structural properties of a given number system. As a trivial example, the division computed in fig.4 is only possible because of the positional properties sketched in sec.1. However, we would like to give a more precise account on this idea in this section for some more complex aspects of the relation entertained between internal and external representations may have been forgotten.

Dimension(s) As shown in sec.1, the notion of base is a central one. Some trivial and primitive systems do not use any distinct base to express numbers above the extension of the base – e.g. the *add one* ■ system sketched in sec.1 or an *add-one-finger* system – whereas most of the systems developed in history across different cultures use a base – most of the time, 10. In Zhang and Norman 1995's vocabulary, such systems like Hindu-Arabic, Egyptian, Greek or Chinese are *two dimensional* (1x1D) for they distinguish the *base dimension* from the *power dimension* (base x power) which allow them to be decomposed in a polynomial fashion ($\sum a_i x^i$ for a the segment value and x the base value) – see eq.4 p.5.

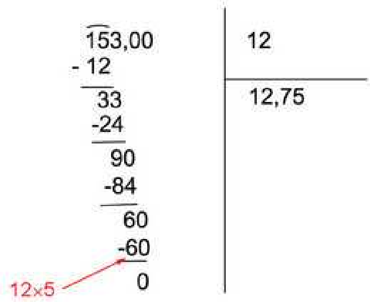

Figure 4: Hand division in Hindu-Arabic system

As we also saw it, some systems as the Babylonian but also the Mayan and the Roman – which is a bit more complex case as we will see later on – systems use two different bases – a main base and a smaller sub-base – which make them *three dimensional* systems ((1x1) x1D). Before going any further, let's take the example

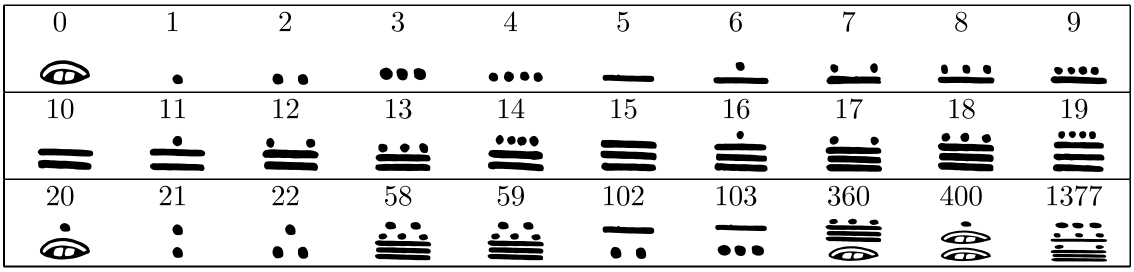

Table 3: Mayan system

of the Mayan system. Up to the limit of the main base, this system looks like any other (1x1D) as tab.3 suggests it. However, above 19, the system starts to behave strangely for someone accustomed to an SPNS such as the Hindu-Arabic system. While the fact that, up to a certain point, a set of "*makes*" a is not surprising for it seems the vertical equivalent of many (1x1D) systems presented in tab.4, the fact that the system *adds one level* every time the limit (20) is reached on the precedent level is more striking. Thus, the power every floor corresponds to $20^{level-1}$. As an example, eq.8 presents a decomposition of the Mayan equivalent for 1377 – see Zhang and Norman 1995:276 for a similar decomposition of the Babylonian System.

3^{rd} floor		$(0 * 5^1 + 3 * 5^0) * 20^2$	=1200
2^{nd} floor		$(1 * 5^1 + 3 * 5^0) * 20^1$	=160
1^{st} floor		$(3 * 5^1 + 2 * 5^0) * 20^0$	=17
			1377

(8)

Representations What have been said above concerns only the *Abstract Structures* or the deep logics of these numbers systems. On a representational level, the fact that two systems have the same *level of dimensionality* does not mean that they

Systems	Example	Base	Base Dimension	Power Dimension
Abstract	$\sum a_i x^i$	x	a_i	x^i
Arabic	447 $4\times10^2+4\times10^1+7\times10^0$	10	a_i = *shapes*	x^i = *positions*
Egyptian	𓍢𓍢𓍢𓎆𓎆𓎆𓎆𓏺𓏺𓏺𓏺𓏺𓏺𓏺 $4\times10^2+4\times10^1+7\times10^0$	10	a_i = *quantity* The number of 𓍢 𓎆 𓏺	x^i = *shapes* 𓍢 𓎆 𓏺 10^2 10^1 10^0
Greek	υμζ $4\times10^2+4\times10^1+7\times10^0$	10	a_i = *shapes* α β γ ....θ 1 2 3 9	x^i = *shapes* ι κ λ .... 1×10^1 2×10^1 3×10^1............ ρ σ τ .... 1×10^2 2×10^2 3×10^2............

Table 4: Some 1x1D systems

Base dimension									**Power dimension**
α	β	γ	δ	ϵ	ϝ	ζ	η	θ	$*10^0$
ι	κ	λ	μ	ν	ξ	o	π	ϙ	$*10^1$
ρ	σ	τ	υ	ϕ	χ	ψ	ω	ϡ	$*10^2$
1	2	3	4	5	6	7	8	9	

Table 5: Power dimensions in the Greek system

are the same for some systems would use different representations of their respective dimensions. Typically, physical counterparts of dimensions could be *Quantities* (Q) – i.e. number of symbols – *Positions* (P) or *Shapes* (S). Thus, while the Hindu-Arabic, the Greek and the Egyptian systems belong to the (1x1D) family, the first would represents the base with a symbol – e.g. "6" – and the power dimension with the rank on a line (SxP) while the others would use only shapes (Greek or Chinese: SxS) or quantities and shapes (Egyptian: QxS) – see tab.4 for details. To highlight this distinction, let's note that, for the Egyptian system, "$\cap\cap\cap\bullet\bullet$ " means the same thing as "$\bullet\bullet\cap\cap\cap$" or even "$\cap\bullet\cap\bullet\cap$" (32) though in a maybe less canonical fashion.

The same remark holds for (SxS) systems such as the Greek system. Even if the Greek alphabet is an ordoned set of symbols, the numerical system using these

Systems	Example (447)	Main Base	sub Base	Sub-base Dimension	Sub-power Dimension	Main-power Dimension
Abstract	$\sum\sum(b_{ij}y^j)x^i$	x	y	b_{ij}	y^j	x^i
Babylonian	∇∇∇∇∇∇∇ ≺≺∇∇∇∇∇∇∇ $(0\times10^1+7\times10^0)60^1$ $+(2\times10^1+7\times10^0)60^0$	60	10	$b_{ij}=quantity$ The number of ∇ ≺	$y^j=shapes$ ∇ 10^0 ≺ 10^1	$x^i=positions$ $60^2..60^1...60^0$
Mayan	• •• $\overset{\bullet\bullet}{\rule{2em}{1pt}}$ $(0\times5^1+1\times5^0)20^2$ $+(0\times5^1+2\times5^0)20^1$ $+(1\times5^1+2\times5^0)20^0$	20	5	$b_{ij}=quantity$ The number of • —	$y^j=shapes$ • 5^0 — 5^1	$x^i=positions$ $20^2..20^1...20^0$
Roman	CCCCXXXXVII $(0\times5^1+4\times5^0)10^2$ $+(0\times5^1+4\times5^0)10^1$ $+(1\times5^1+2\times5^0)10^0$	10	5	$b_{ij}=quantity$ The number of I, V, X etc	$y^j=shapes$ $I=10^0\times5^0$ $V=10^0\times5^1$ $X=10^1\times5^0$ $L=10^1\times5^1$	$x^i=shapes$ $I=10^0\times5^0$ $V=10^1\times5^0$ $X=10^1\times5^0$ $V=10^1\times5^1$

Table 6: Some (1x1)x1D systems

symbols does not behave as a SPNS for the power is represented by subsets of this set of symbols defined by the alphabetical order. This point implies that $\sigma\lambda\beta$ $(2*10^2+3*10^1+2*10^0=232)$ also has the same meaning as $\lambda\beta\sigma$ $(2*10^2+3*10^1+2*10^0=232)$ even if, for practical reasons, symbols were ranked by power. (1x1)x1D systems are also susceptible to represent their three dimensions in various ways as tab.6 suggests it. To take our precedent example of the Mayan system, the base is represented by the *quantity* of •••, the first power by shape's difference – ••• vs — – and the main power by positions (levels) – for more examples, see Zhang and Norman 1995:279 or Ifrah 1987.

These technical aspects highlighted, the question of whether and how this representational level will interact with the effective computation can be raised in appropriate words. Here Zhang and Norman 1995 propose to investigate this question with the same framework presented in sec.3.1 suggesting that systems differ from each other in two main aspects, namely, i) the faculties used for identification and the ii) externality of the representation of the relevant structural informations.

Faculties While *Shapes* type of representations suggests a merely recognitional faculty, *Quantities* and *Positions* involve different visual data type of which, say symmetry, could interact with some operations. Though this first aspect may seem a trivial one, it is quite obvious that, for different bases such as the 20-based Aztec system, the fact that *bases* are represented by *Quantities* certainly has a cognitive impact for it seems hard to distinguish ●●●●●●●●●●●● (12) from ●●●●●●●●●●●●● (13) in one gaze. However, this distinction could seem a bit sharp for one could easily considers that symbols such as ••• or  implies a recognitional faculty similar to the one used for, say, "2" and "7". We will go back on this issue in sec.4.1.

Externality Besides this first superficial aspect, types of representations may or may not represent externally the *properties* – *category*, *magnitude*, *equal interval* and *absolute zero* – associated with the *psychological scales* – ratio, interval, ordinal and nominal – qualified in Stevens 1946. To be more specific, whereas nominal scales – e.g. a list of names – only have the formal property to distinguish their elements (*category*), ordinal scales – e.g. rank of participants in a race – also have an ordoned structure (*category* and *magnitude*). Adding *equal intervals* – e.g. there is the same amount of time between 8AM and 10AM and 10AM and 12AM – to any ordinal scale makes it a interval scale which only lack an *absolute zero* to behave like a ratio scale – e.g. it does not have any sense to ask whether 10AM is twice as late as 5AM. Thus ratio scales possess all the four formal characteristic properties of numbers *per se*. However, numbers systems do not spatially implement these formal characteristics in the same way.

The base and power dimensions of all numeration systems are abstract

> dimensions with ratio scales. In different numeration systems, these abstract ratio dimensions are implemented by different physical dimensions with their own scale types. (Zhang and Norman 1995:281)

To clarify this assertion, whereas *Shapes*, as a *categorical* formal property, involves to represent all the other formal properties of number relations – e.g. ”x is next y” (*magnitude*) or ”x is twice y” (*equal interval* and *absolute zero*) – in an internal manner, the *Quantity* implementation of base dimension in the Egyptian system makes these features directly accessible without any resort of memory. In other words, the fact that ”||” is the next number after ”|” and that ”||||” is twice ”||” are immediately visible as physical features in such a number system.

As soon as we consider more complex cases, typically two-dimensional systems above base, interaction between the representations of dimensions makes interesting predictions about tasks such as comparison and operations – see sec.3.3 – across the different numerical systems. As an example, if both base and power dimensions are represented by ordinal or more complex scales, all the information needed for a magnitude comparison of two numbers is external which is not the case for *category*-based systems such as the Greek Chinese – see fig.4 – or Hebrew systems. As we saw it for the greek system, if one wants to know if $\phi\gamma$ is higher than $\upsilon\mu\alpha$, she should first identify the symbol coding for the higher power – here ϕ and υ – to know that $\phi\gamma$ (503) is higher than $\upsilon\mu\alpha$ (441). Such a procedure depends massively on memory which is not the case for a (SxP) systems as the Hindu-Arabic in which positions make immediately visible that 441 is higher than 53. To conclude, representational effects depends on both internal and external representation but also on the interaction of these representations with the given dimension they represent.

Rather than giving a complete account on the way different types of representations across the different dimensions of a given system may interact with various tasks, we would like to suggest this diversity with the example of multiplications.

3.3 Impact on calculation

Three levels in multiplication On an abstract level, a multiplication can be realized in different ways. From addition iteration – see eq.9 – to polynomial decomposition – see eq.10 and eq.11 for an example – the level described here corresponds to the *algebraic level* properties of which are the direct consequence of the properties of the division ring $< \mathbb{Z}, \leq, +, *, 0, 1 >$.

$$N1 * N2 = \overbrace{\text{n1+n1+...+n1}}^{\text{n2 time}} \tag{9}$$

$$\begin{aligned} N1 * N2 &= \sum a_i x^i * \sum a_j x^j = \sum \sum a_i x^i a_j x^j \\ &= \sum \sum a_i a_j x^{i+j} \end{aligned} \tag{10}$$

$$\begin{aligned} 43 * 35 &= (4 * 3) * 10^{1+1} + (3 * 3) * 10^{0+1} + (4 * 5) * 10^{1+0} + (3 * 5) * 10^{0+0} \\ &= 40 * 30 + 3 * 30 + 40 * 5 + 3 * 5 \\ &= 1200 + 90 + 200 + 15 = 1505 \end{aligned} \tag{11}$$

However, while we could think that polynomial decomposition constitutes some king of algorithmic process, authors argue that "*[a]t the level of algorithms, different algorithms can be applied to the same algebraic structure*" (Zhang and Norman

1995:285). Further, the polynomial structure would be the only genuine – i.e. non-reducible to any additive process – multiplicative structure. Finally, "*For all 1x1D systems, term multiplication ($a_i x^i * b_j y^j$) has the same set of six basic steps [i.e. algorithm level]*" (Zhang and Norman 1995:286) thus numeral system differ only toward the *level of number representation*.

Algorithms for polynomial multiplications over ER Table 7 reproduce in a synthetic manner the decomposition of the different steps of an algorithm. For every single number system, the externality or internality induced by the different representations – *Quantities*, *Shapes* or *Positions* – involved for every step is printed in bold font. As a general observation, while the calculation step 2b is merely internal for all these numbers systems, some others representationally-dependent steps differentiate distinctly these systems and, more specifically, the Hindu-Arabic positional notation of which enable the computing subject to base the power steps – 3b and 4 – solely on external representations.

Notwithstanding, the reader may oppose that the special status of the Hindu-Arabic system is the direct consequence of the choice of the algorithm uniqueness of which is dubious across these different systems. Steps 2b and 4 seem especially questionable in the case of the Greek and Egyptian systems for they give the impression of an isomorphic positional-based calculus despite the fact that, as shown above, (SxS) and (QxS) systems behave very differently than (SxP) systems. In other words, things are presented as if Quantities of • and Shapes were computed following the rules of a positional based system before a miracle occurs – $*10^1$ on step 4.

However, Zhang and Norman 1995 never claimed that the algorithm was unique

Step	Abstract	Greek ($\lambda * \delta$)	Egyptian ($\cap\cap\cap * \vert\vert\vert\vert$)	Arabic ($30 * 4$)
1	separate power from base	**internal**	**external**	**external**
2a	get base values of $a_i x^i * b_j y^j$ $B(a_i x^i) = a_i$ $B(b_j y^j) = b_j$	**internal** (S) $B(\lambda) = \gamma B(\delta) = \delta$ $B(\delta) = \delta$	**external** (Q) $B(\cap\cap\cap) = \vert\vert\vert$ $B(\vert\vert\vert\vert) = \vert\vert\vert\vert$	**internal** (S) $B(30) = 3$ $B(4) = 4$
2b	multiply base values $a_i * b_j = c_{ij}$	**internal** (mult. tab) $\gamma * \delta = \iota\beta$	**internal** (mult. tab) $\vert\vert\vert\vert * \vert\vert\vert = \cap\vert\vert$	**internal** (mult. tab) $3 * 4 = 12$
3a	get power values of $a_i x^i * b_j y^j$ $P(a_i x^i) = i$ $P(b_j y^j) = j$	**internal** (S) $P(\lambda) = 1$ $P(\delta) = 0$	**internal** (S) $P(\cap\cap\cap) = 1$ $P(\vert\vert\vert\vert) = 0$	**external** (P) $P(30) = 1$ $P(4) = 0$
3b	add power values $P(a_i x^i b_j y^j) = 1 + 0$	**internal** (add. tab) $P(\lambda * \delta) = 1 + 0$	**internal** (add. tab) $P(\cap\cap\cap * \vert\vert\vert\vert) = 1 + 0$	**external** (S) $P(30 * 4) = 1 + 0$
4	attach power values $a_i x^i * b_j y^j = c_{ij} * x^P ij$	**internal** (S) $\iota\beta * 10^1 = \rho\kappa$	**internal** (S) $\cap\vert\vert * 10^1 =$ 𓍢 $\cap\cap$	**external** (S) $12 * 10^1 = 120$

Table 7: Steps implementations across various 1x1D systems

for different algorithms may use these six basic steps in different orders and group partial products in a different fashion than a contemporaneous westerner. As an example, eq.12 presents an Hindu-Arabic algorithm which follows the same order than tab.7 taking first the base values of N1 and N1 (2a) before adding the power values (3b). In contrast, the Greek algorithm presented in eq.13 groups the partial products in a different way inasmuch as it attaches first the power values to bases before adding the results. In other words, even if the six steps are necessary to perform a calculation, the order, induced by the specificities of a given numbers system, can change with some restriction to make a system-specific algorithm easier to manipulate in the given number system.

Abstract Form		Greek System	Arabic System
a_1x^1	a_0x^0	$\iota \quad \zeta$	1 7
b_1x^1	b_0x^0	$\iota \quad \gamma$	1 3
	$a_0b_0x^0$	$\kappa \quad \alpha$	2 1
$a_1b_0x^1$		λ	3
$a_0b_1x^1$		o	7
$a_1b_1x^2$		ρ	1
$a_1b_1x^2 + (a_1b_0x^1 + a_0b_1x^1) + a_0b_0x^0$		$\sigma \quad \kappa \quad \alpha$	2 2 1

(12)

What makes the Hindu-Arabic system so special? As a conclusion of this section, we could ask whether the Hindu-Arabic is special in the strict sense induced sometimes with the idea that mathematics would have been impossible without such a system. As shown above, the Hindu-Arabic system is not the only positional system for some other systems also use positions to represent the power dimension – see fig.6. However, it is indeed the only (SxP) system among 1x1D systems specificity of which is responsible for the algorithm used to perform some calculations such as multiplication. Once again, this does not mean that other methods are impossible – see eq.13 for a counterexample – nor does it mean that such methods are less effective in general. Even if the Arabic system is more efficient for large numbers and that its multiplication tables are smaller than the one required for, say, the Aztec system (base =20), "*the abacus, which was still widely used in Japan, China,* *and Russia before electronic calculators became popular, is more efficient than the Arabic system for addition and subtraction*" (Zhang and Norman 1995:292).

To summarize, the representational effect of numbers systems has to do with the way the externality (resp. internality) of the representations of the different dimensions interacts with mathematical tasks. From magnitude comparison to multiplication, the way properties of numbers are visually encoded will give rise to an algorithm specific to the given task and number system or, in other words, a specific correspondence of external rules or constraints towards external representations with their internal counterparts.

Abstract Form		Greek System	Arabic System
a_1x^1	a_0x^0	$\iota \quad \zeta$	10 7
b_1x^1	b_0x^0	$\iota \quad \gamma$	10 3
$a_1b_1x^2$	$a_1b_0x^1$	$\rho \quad \lambda$	100 30
$a_0b_1x^1$	$a_0b_0x^0$	$o \quad \kappa\alpha$	70 21
$(a_1b_1x^2 + a_0b_1x^1) + (a_1b_0x^1 + a_0b_0x^0)$		$\rho o \quad \upsilon\alpha \quad \sigma\kappa\alpha$	170 51 221

(13)

4 When problem solving finally discuss with neurology

How could we possibly compare these two very different cognitive approaches of mathematics ? Given that the first one –sec.2– endorse an internalist and numerosity-based conception of number representation which relies on neural effects of low level tasks and that the second one –sec.3– suggests a distributed representations paradigm based on a problem-solving methodology, it is not surprising that these two framework did not discuss directly before Zhang and Wang 2005. In this section, we would like to suggest that the direct question to whether numbers are or are not internally represented as continua or as format-independent analog representations is not a good one while the *scale problem* is not disentangled. We will first underline the weaknesses of both frameworks – sec.4.1 and sec.4.2 – to suggest in a third time how these frameworks could benefits from each other.

4.1 Some problems with the Distributed Representations framework

Besides the extreme generality of a framework used in many context from usability criterion for graphic user interfaces (Zhang 1996) to medical errors (Zhang, Patel, et al. 2004), Zhang and Norman 1995 lack a criterion for say "*entities*" from other visual representations. As an example, *quantities* are said to differ from *shapes* type of representations. This difference is crucial for it justify the idea that a (SxP) system such as the Hindu-Arabic does not represent the base dimension in the same way that a (SxP) system such as the Egyptian. More generally, a similar criterion is applied for all the nomenclature proposed in tab.4 p.20 and tab.6 p.21. However, when the time to distinguish external from external representations of formal properties comes, the difference of ||| from, say, || (*category*) becomes immediately available which seems to imply a recognitional faculty very similar that the one used for *Shapes* as we pointed it out in sec.3.2. Therefore, given the fact that the cognitive taxonomy proposed in tab.4 p.20 and tab.6 p.21 contradicts what is said about the externality of the *category* property, the criterion for say *entities* such as • from combinations of these (say ••) is merely a graphical and syntactical one – very similar with the one proposed in sec.1 in fact – which relies on key assumptions concerning deep logical structures of numerical systems.

Furthermore, even 1x1D and (1x1)x1D systems are not easily distinguishable from a graphical components analysis. One could easily oppose that putting apart deep logical structures of number systems on one hand and the representations of such dimensions on the other before linking them is illusory or circular given the fact that the distinction of dimensions presuppose some assumptions about the behavior of the graphical components. To go back on our example, instead of a (QxS)xP system, the Mayan system could be considered as (SxP) 20-based system provided that •••• — $\dot{\text{—}}$ are interpreted as symbols or digits similar with the Hindu-Arabic system (4,5,6). In other word, what does assure that this system is not more coarse-grained except an intuition that •••• and — are, contrary $\dot{\text{—}}$, to the primitive (cognitive) components of such a system.

4.2 Issues with the Neuro-internalist approach

On the other hand, the internalist conception endorsed by most of the cognitive-neuroscience part of the literature suffers from opposite problems for i) it relies on deep assumptions about the necessity of encoding before effective computation (see cit. p.11) ii) supported by low level tasks such as digits comparison which iii) use different notations only in trivial cases.

First of all, we should notice that these problems have been putted under lights from a part of the cognitive-neuroscience community itself. In this regard, variations of typical tasks used in this fields to induce mathematical-related brain activities together with more precise brain-imaging technics than fMRI (anatomical resolution of 3x3x3 mm) such as fMRA allowed Cohen Kadosh et al. 2007 to propose clusters sensitive to certain notations. Though, we consider that the issue goes beyond the notation (in)dependent or (non)symbolic issue for, as we suggested it in sec.3.3, magnitude comparison across notations involve very specific mechanisms different from those of calculation. Furthermore, we also saw that systems differ mainly above the power base while comparisons often proposed by this literature concern numbers below 10. In other words, it is not surprising that •• and || may rise the same *numerosity signal* provided that the base dimension in encoded with *Quantities* and that the non-quantitative encoding –e.g. "TWO" and "2"– are familiar. Finally, given that cardinality is said to be the only relevant external feature extracted from sets (like in a 1D system), this approach cannot account for the immediate use of the structural properties of notations which are the *sine qua none* conditions to represent numbers above the cognitive limits of discrimination – see fig.5.

What does append beyond the base? As a first direct discussion of the two frameworks Zhang and Wang 2005 missed a decisive aspect of the debate inasmuch as they choose to ignore the fact that most of the studies in this field do not deal with complex tasks –a.k.a. *scale problem*– to ask directly whether the so-called sequential model endorsed by Dehaene 1992; Dehaene, Dupoux, and Mehler 1990 is valid in number comparison paradigm. However, the results for variants of the classical two-digits number comparison presented in Zhang and Wang 2005 suggest that some of the internalist neuro-scientists who investigate these mathematical tasks did not take in consideration any representational effect. Experiments presented in Dehaene, Dupoux, and Mehler 1990; Hinrichs, Berie, and Mosell 1982 consisted in saying whether some two-digits numbers were above or below 55. Yet, changing three parameters – i) presenting simultaneously two targets with the standard ii) using two different standard (55 and 65) and iii) ask the subject which target is smaller (resp. larger in exp.2) rather than if a sole target is smaller (resp. larger) than the standard – produces some effects presence of which across the different situations is not predictable without representational effects; namely

Unit-effect the fact that numerals within the same decade (e.g. 31–39) have different RTs

Decade effect the fact that the mean RTS of different decades (e.g. 31–39 and 61–69) are different

Discontinuity effect which consists in sharp changes in RTs across the boundaries

Strop-like effect the fact that unit digits may interfere or facilitate the comparison of two digits numeral (e.g. the 1 of 21 could facilitate the comparison while the 9 of 29 interfere with it)

Without detailing the weighted results across conditions, the difference in standards (55 and 65) together with the presence (resp. absence) of the effects advocates for a huge effect of external representations. However, the very experimental paradigm adopted in Zhang and Wang 2005 – i.e. the fact that standards were presented simultaneously with targets rather than memorized – cannot judge whether merely internal representations are notation-specific for their experiment uses direct perceptions of numbers. In other words, the tasks are not the same thus do not test the same faculties. We would like to disentangle this deadlock in the next subsection.

4.3 How to solve the Scale Problem: some proposals

It seems obvious that, for large numbers or complex operations, the mind will relies on specialized representations in accordance with the number system he uses generally – see sec.3.3. The idea that representational differences constraint the development of algorithms specific to a given number system – see eq.13 p.28 – is also parsimonious enough to be endorsed.

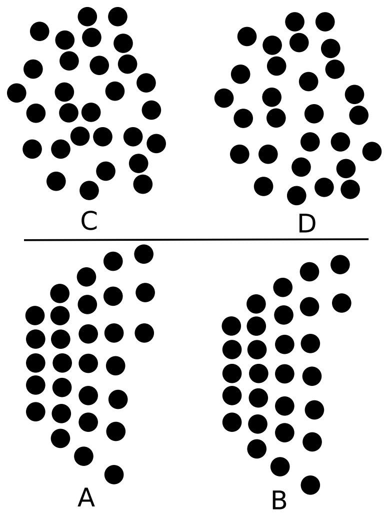

Figure 5: Cardinality vs structural properties ($Card(A) = Card(C)$, $Card(B) = Card(D)$ and $Card(A) - Card(B) = Card(C) - Card(D) = 1$)

> For example, the studies by Gonzalez and Kolers 1982, 1987 suggest that Arabic and Roman numerals were processed differently. They argue that during numerical processing people do not transform different external number representations into a common abstract internal representation. Rather, they operate upon different internal representations that reflect the physical characteristics of different external representations.(Zhang and Wang 2005:832)

However what is not so clear is whether such a spatial and notation-specific representation is required for more easy tasks and from which degree of complexity it might be so. In other words, what is not sure is whether notations matter when we solve 3-1. Here is what can be labeled as the *scale problem* which happens on both part of the debate. For the Zhang and Norman 1995'framework, it concerns the distinction of quantities from shapes, as we saw in sec.4.1, for it seems possible to consider that  is a symbol just like ”6”. On the other side, the internalist account of Dehaene, Dehaene-Lambertz, and Cohen 1998 *plays at an other level* inasmuch as the putative compulsory encoding for computation i) concerns mainly numbers below 10 ii) which are, so to speak, thought and not *visualized* – externaly or internaly – like in a complex operations.

Thus, given the fact that the former deals with ER-based complex operations without being able to give a precise account on units and that, conversely, the later is mainly concerned with internal representation simplicity of which do not have anything to do with algorithmic manipulations of numbers, it would not be excessive to suggest that these frameworks do not contradict each other and might be complementary. As an example, we saw in sec.2 that the posterior superior parietal lobule (PSPL) is considered to be involved in dual tasks. Thus, given that some operations do require such dual tasks – e.g. keep 3 to add it to the product of 2*6 for 25*6 – and that such operations do depend on the given number system, the possibility that numerosity-based computation works together with a more distributive tasks system algorithms of which are sensitive to external representations is still open as an empirical inquiry.

New account on the SNARC effect Originally discovered by Dehaene, Bossini, and Giraux 1993, the Spacial-Numerical Association of Response Code effect have been invoked to support the idea that values or numerosities are linearly ordonned on a mental line inasmuch as, no matter how irrelevant the magnitude can be in tasks such as odd–even classifications, response times are faster with left hand for small digits (e.g. 2) and with right hand for large digits (e.g. 9). This interaction of space and number was the only interaction of numbers representations with space endorsed by these authors.

Though, current research on the SNARC effect suggests that this effect might be dependent on cultural diversity – Cf the case of Iranian and Japanese subjects in Hubbard et al. 2005 – and that two-digit number behave differently (Nuerk, Weger, and Willmes 2001; Nuerk and Willmes 2005), suggesting that ”[...] *the mental representation of two-digit numbers depended on the stage of processing*” (Zhou et al. 2008:1535)

> The initial processing seemed to involve the compositions of the two-digit numbers as shown by the significant distance and magnitude effects for unit digits in Experiment 2 (simultaneous presentation). Once the numbers are stored in the short-term memory, however, they showed holistic representation, as shown by a lack of distance and magnitude effects for unit digits in Experiment 1 (serial presentation).(Zhou et al. 2008:1535)

This quotation is interesting for three reasons. It first insists on the impact of presentation modes – simultaneous or serial – thus on the fact that Dehaene, Bossini, and Giraux 1993 and Zhang and Wang 2005 do not investigate the same mechanisms – see also Ganor-Stern, Pinhas, and Tzelgov 2009. Further, the idea that a compositional representation of multi-digits numbers is distributed across different levels of processing – see. Notebaert et al. 2006 for similar results – might give an account on the *scale problem* inasmuch as the question of whether the different dimensions of a given number system correspond to distinct representational processes becomes an empirical opened question – see Wood, Nuerk, and Willmes 2006 for an fMRI manifestation of these levels. To take our previous example, given that one can dissociate simultaneous and serial presentation for many kind of numerals, it would be empirically possible to decide whether  is a composed representational component distinct from more primitive ones or a primitive symbol for the given cognitive subject.

Finally, such an impact of the visual field on number representations suggests that the positional nature of the Hindu-Arabic system do interfere with the way these numbers are internally represented and analyzed. To sum up, while representational effect in mathematics was only endorsed by the problem-solving fields during the 90', this question tends to be raised in many different scientific communities from embodied problem-solving (Domahs et al. 2010; Öllinger, Jones, and Knoblich 2008) to neuro-imaging (Liu et al. 2006; Pesenti et al. 2000). Even if more oriented toward the Hindu-Arabic based representations, all the methodological and theoretical tools required to give a precise account on distinct influences of space – i.e. representations – on internal representations of numbers across different numeral systems are now endorsed by the relevant cognitive fields.

Why do not we use abstract grammars? In this last paragraph, we would like to put forward an experimental paradigm which haven't been endorsed by this field despite its obvious relevance. Abstract grammars are sets of syntactical rules which describe the set of possible chunks of a given language. First popularized by Reber 1967, the use of abstract grammars in experimental investigation of learning mechanisms has shown interesting sensitivity for properties such as symmetry of chunks (Jiang et al. 2012; Mealor and Dienes 2012).

We suggest that using the subjective acceptability of chunks generated from abstract grammar could be a good way to investigate the cognitive effect of visual features for i) such grammars would be isomorphic with existent systems while ii) they do not need strong assumptions about the dimensional structures of existing numbers systems iii) and do not induce any surface bias consecutive from previous Hindu-Arabic knowledge. As an illustration of the kind of chunks the subject could be presented during training, given what has been said in sec.3, eq.14 is isomorphic with a subtraction in a (QxS) system – such as the Egyptian – while eq.15 and eq.15 match respectively with multiplication in (SxP) system – such as the Hindu-Arabic – and addition in a (SxS) system – such as the Greek.

$$\clubsuit\clubsuit\clubsuit\clubsuit\blacksquare\blacktriangle\bigstar \ltimes \bigstar\clubsuit = \clubsuit\clubsuit\clubsuit\blacktriangle\blacksquare \qquad (4111 - 1001 = 3110) \tag{14}$$

$$\boxdot \boxminus \boxminus \rtimes \boxminus = \boxminus \boxplus \boxplus \qquad (122 * 2 = 244) \tag{15}$$

$$ak \dotplus ld = me \qquad (11 + 24 = 35) \tag{16}$$

Conclusion

By confronting two traditions and their respective account on the cognitive dimension of mathematics (sec.2 and sec.3), we suggested that the naive question of whether numbers internal representations are notation independent does not have any sense if asked abstractly. As we saw it in sec.4, such a question depends on the level of complexity together with the external availability of numbers during the given task.

To sum up, representational effects of numbers systems are obvious for complex tasks and do depend on the external availability of various properties of numbers.

While this idea has exclusively been endorsed by the problem solving fields since the 90', most of the neuro-scientific part of the research focused on low level tasks supporting a merely internalist conception of numbers representations. Yet, such an antagonism together with the absence of any direct discussion of these frameworks do not mean that they contradict each other for i) hybrid models are nowadays endorsed by a part of the literature (Zhou et al. 2008) ii) which seems able to provide empirical evidence for distinguishing base from power dimensions.

As said in the end of sec.1, it does not seem plausible that a human subject scan every digit following syntactical rules. However, the idea that numbers perception and manipulation do require compositional structures for more primitive components is quite obvious. What remains an empirical issue concerns the level of complexity of theses primitive components. Finally, even if the majority of this literature focuses exclusively on the representational effects induced by SPNS such as the Hindu-Arabic, research on the impact of different spacial properties of n-SPNS is easily conceivable and, actually, endorsed by some teams guided by the idea that algorithms are sensitive to such visual properties (Domahs et al. 2010; Öllinger, Jones, and Knoblich 2008). As a final word, we would like to highlight the indispensability of similar accounts on n-SPNS effects for they appear the best way to contrast the cognitive particularities induced by SPNS.

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